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PRICE-QUANTITY ADJUSTMENTS IN MULTIPLE MARKETS WITH RISING DEMANDS[1]

Kenneth J. Arrow

1. Introduction

In the classical account of the Law of Supply and Demand, it is assumed that the price on any market moves upward if demand exceeds supply and downwards in the opposite case. The discrepancy between supply and demand is assumed to have a real, if transitory, significance. Thus a sudden upward shift in the demand curve gives rise to an excess demand or "shortage," which in turn causes a rise in price which eventually wipes it out.[2] From the point of view of market behavior a shortage manifests itself as unfilled orders. In the case of a labor market, this means unfilled vacancies; that is, firms are willing to hire more workers than they can find at the wage they are currently paying.

In another study,[3] it has been suggested than an extension of this analysis explains some aspects of the observed shortage of engineers and scientists in the United States over the greater part of the post-war

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1. This paper is part of an economic analysis of the engineer-scientist market conducted for The RAND Corporation in collaboration with A. Alchian and W. Capron.

2. See, e.g., Marshall's account of the path to equilibrium on a corn market, A. Marshall, Principles of Economics, Eighth Edition, New York: Macmillan, 1948, pp. 332-334. For the interpretation of the Law of Supply and Demand in interrelated markets, see P. A. Samuelson, Foundations of Economic Analysis, Cambridge, Mass.: Harvard University Press, pp. 270-273.

3. A. A. Alchian, K. J. Arrow, W. M. Capron, "An Economic Analysis of the Market for Scientists and Engineers," The RAND Corporation, RM-2190-RC, 6 June 1958.

period. If the demand curve for a commodity, in this case the services of engineers and scientists, is rising over time, the equilibrating effects of the rising price can be offset by the rise in the demand curve, which raises the equilibrium price as fast as the market price moves towards it. A shortage, in the sense of an excess of demand over supply, may therefore persist for an extended period of time during which the market price is steadily rising.

In the present paper, account is taken of the fact that there is not one market for engineers and scientists but many interrelated ones. There are many different types, such as chemical, mechanical, and civil engineers, mathematicians, physicists, and many others. Within each type, there are many grades of ability and specialty. To a considerable extent, these different types and grades are substitutes for each other either on the demand or the supply side or both. Thus, many engineers can perform the functions of mathematicians or physicists, or, if theoretical mathematicians are expensive, they can to a certain extent be replaced by increased computing, which in turn leads to demands for electrical engineers as well as less qualified mathematicians.

The model, which will be formulated abstractly, then consists of a set of markets. On each one the demand is a function of all prices. a rise in the price on any market decreases the demand in that market but increases the demand in some or all other markets, since all the commodities are substitutes. Similarly, the supply on each market is a function of all prices; a rise in any one price increases supply on that market but decreases it on all other markets, since it draws the supply to that market. It is assumed that the price on each market

obeys the Law of Supply and Demand. In the absence of disturbances, the prices will eventually approach equilibrium, with equality of supply and demand on all markets.

The paper studies the effect of steady upward shifts in some or all demand functions on the adjustment path of all these prices and quantities. It is shown that the shortage on each market increases to a limiting value, while prices rise with the rate of increase which itself increases to a limiting value depending only on supply and demand conditions.

2. Interrelated Markets in the Absence of Trends

We suppose there are n services or commodities which can be supplied from the same or related sources and which are demanded by the same or related industries. Let p_i $(i = 1,\ldots,n)$ be the price of the service or commodity on the ith market, and let S_i be the supply forthcoming at any moment of time. The supply of any one commodity will depend not only on its price but also on the prices of all other commodities, since higher prices on other markets will draw supplies away from the given market. We can thus write,

$$(1) \qquad S_i = S_i(p_1,\ldots,p_n), \text{ with } \partial S_i/\partial p_i > 0, \ \partial S_i/\partial p_j \leq 0 \text{ for } i \neq j.$$

In a linear approximation, we can write,

$$(2) \qquad S_i = \sum_{j=1}^{n} a_{ij}p_j + c_i \quad (i = 1,\ldots,n),$$

with

$$(3) \qquad a_{ii} > 0, \ a_{ij} \leq 0 \text{ for } i \neq j.$$

Similarly the demands for the different commodities may be interrelated: an increase in the price of one commodity will cause demand to shift from it to other commodities which are substitutes for it.

(4) $D_i = D_i(p_1,\ldots,p_n)$ with $\partial D_i/\partial p_i < 0$, $\partial D_i/\partial p_j \geq 0$ for $i \neq j$.

Because of the weak inequalities in the last clause, the case where all demands are independent, that is, D_i depends only on p_i, is included as a special case. As a linear approximation to (4), we have,

(5) $D_i = \sum_{j=1}^{n} b_{ij}p_j + d_i$, $b_{ii} < 0$, $b_{ij} \geq 0$ for $i \neq j$.

On each market, there will be usually a "shortage," i.e., the difference between demand and supply, which we will denote by X_i.

(6) $X_i = D_i - S_i$.

(Of course, X_i might be negative, in which case there is a "surplus.")

We assume that on each market, the price moves as directed by the shortage X_i, rising if the shortage X_i is positive, decreasing if negative, and remaining stationary if zero. To a linear approximation,

(7) $dp_i/dt = k_iX_i$, $k_i > 0$ $(i = 1,\ldots,n)$.

Substitute from (2) and (5) into (6).

(8) $X_i = \sum_{j=1}^{n} (b_{ij} - a_{ij})\, p_j + (d_i - c_i)$ $(i = 1,\ldots,n)$.

For convenience, let,

(9) $m_{ij} = b_{ij} - a_{ij}$ $(i \neq j)$, $m_i = d_i - c_i$.

Then (8) can be written,

$$(10) \qquad X_i = \sum_{j=1}^{n} m_{ij}p_j + n_i \quad (i = 1,\ldots,n).$$

From (9), (3), and (5), we see that,

$$(11) \qquad m_{ii} < 0, \; m_{ij} \geq 0 \text{ for } i \neq j.$$

The equilibrium situation is one of equality of supply and demand on all markets, that is, $X_i = 0$ for all i. Then (10) yields a system of linear equations which can be solved for the equilibrium price. The approach to equilibrium is described by equations (7) and (10). These combine to yield,

$$(12) \qquad dp_i/dt = k_i \sum_{j=1}^{n} m_{ij}p_j + k_i n_i \quad (i = 1,\ldots,n).$$

Equation (12) constitutes a system of simultaneous differential equations whose solution yields the time paths for each price. We will assume that the system is stable, that is, that each price approaches its equilibrium value.[4]

Let us rewrite the above in vector notation. Let $\underline{x}$ be the vector with components X_i, $\underline{p}$ the vector with components p_i, K the matrix with diagonal elements k_i and off-diagonal elements 0, M the matrix with elements m_{ij} and $\underline{n}$ the vector with components n_i. Then (7), (10), and (12) can be written,

$$(13) \qquad d\underline{p}/dt = K\underline{x},$$

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4. Under the assumptions stated, it can be shown that stability follows from the usual competitive conditions; see F. H. Hahn, "Gross Substitutes and the Dynamic Stability of General Equilibrium," Econometrica, Vol. 26 (1958), pp. 167-170; K. J. Arrow and L. Hurwicz, "On the Stability of Competitive Equilibrium I," Econometrica (forthcoming), Part III, especially Theorem 9.

(14) $X = Mp + n,$

(15) $dp/dt = KMp + Kn.$

We have assumed that the system (15) is stable. The stability depends only on the matrix of coefficients of p, so that we will also say that KM is stable matrix, which we will define as the matrix of a stable system of differential equations (with constant coefficients). Equivalently, a stable matrix is one whose characteristic roots all have negative real parts. Also a matrix with the properties (11), i.e., negative diagonal and non-negative off-diagonal elements, will be termed a Metzler matrix.[5] Since the elements of KM are $k_i m_{ij}$, it follows from (7) and (11) that KM is also a Metzler matrix.

(16) KM is a stable Metzler matrix.

3. Some Mathematical Properties of Stable Metzler Matrices

As a preliminary to the subsequent analysis, we will need some mathematical properties of stable Metzler matrices. Note that for any matrix A we can choose a constant s so that $s + a_{ii} > 0$ for all i; then if A is a Metzler matrix, $sI + A$ has only non-negative elements, where I is the unit matrix. Such non-negative matrices have a number of convenient properties which will be used.[6]

Lemma 1. A principal minor of a stable Metzler matrix is a stable Metzler matrix.

Proof: Let A be a principal minor of the stable Metzler matrix B; obviously A is a Metzler matrix. Choose s so that $sI + B$ is non-negative.

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5. L. Metzler, "Stability of Multiple Markets: the Hicks Conditions," Econometrica, Vol. 13 (1945), pp. 277-292.

6. These properties are collected conveniently in G. Debreu and I. N. Herstein, "Nonnegative Square Matrices," Econometrica, Vol. 21 (1953), pp. 597-607.

The characteristic roots of $sI + B$ are s larger than those of B; since, by hypothesis, the real part of any characteristic root of B is negative, it follows that the real part of any characteristic root of $sI + B$ must be less than s. For any non-negative matrix, there is a real characteristic root λ_o such that $|\lambda| \leq \lambda_o$ for all characteristic roots λ. Let λ_o be this root for the non-negative matrix $sI + B$ and λ_1 the root for the non-negative matrix $sI + A$, which is a principal minor of $sI + B$. Then (see Debreu-Herstein, p. 600, last two lines),

$$(17) \qquad \lambda_1 \leq \lambda_o < s.$$

If λ is any characteristic root of $sI + A$, then $|\lambda| \leq \lambda_1$; since the real part of λ is necessarily not greater than $|\lambda|$, it follows from (17) that the real part of λ is less than s. Since the characteristic roots of A are s smaller than those of $sI + A$, their real parts are all negative, so that A is stable.

Lemma 2. If A is a stable Metzler matrix, then $Ax < 0$ implies $x > 0$ and $Ax \leq 0$ implies $x \geq 0$.

Proof: Choose s as before so that $sI + A$ is non-negative; let $B = sI + A$. Let $y = -Ax > 0$. The characteristic roots of B are less than s in absolute value by the same argument as in the proof of Lemma 1. Then (Debreu-Herstein, Theorem III*, p. 601),

$$(sI - B)^{-1} \geq 0.$$

Since $(sI - B)^{-1}$ is non-singular, each row must have at least one non-zero and therefore positive element. Since $y > 0$,

$$(sI - B)^{-1} y > 0.$$

If we substitute for B and y, we find that $x > 0$. The second part of the lemma follows by continuity.

Lemma 3. If A is a stable Metzler matrix, then $-A^{-1}$ is non-negative.

Proof: Let x be the ith column of $-A^{-1}$; then the vector Ax has -1 in the ith place and 0 elsewhere, by definition of an inverse. Hence $Ax \leq 0$, so that $x \geq 0$ by Lemma 2. Since $\underline{x}$ is any column, the lemma holds.

Theorem. If A is a stable Metzler matrix, $b \geq 0$, $y(0) \geq 0$, and $dy/dt = Ay + b$, then $y(t) \geq 0$ for all $t \geq 0$.

Proof: Suppose the set of times $\underline{t}$ such that $t \geq 0$, $y_j(t) < 0$ for some j is non-null. Let t_o be the greatest lower bound of such t-values. If $t_o > 0$, then by definition $y(t) \geq 0$ for $t < t_o$, so that, by continuity,

$$y(t_o) \geq 0. \tag{18}$$

If $t_o = 0$, then (18) holds by hypothesis.

The functions $y_j(t)$, $\dot{y}_j(t)$ ($= dy_j/dt$) are analytic. For each there is some open interval beginning with t_o in which they have constant sign. By choosing $t_1 > t_o$ but sufficiently close,

$$y_j(t),\ \dot{y}_j(t) \text{ have constant sign over the open interval } (t_o, t_1). \tag{19}$$

Let S be the set of indices $\underline{j}$ such that $y_j(t) < 0$ in (t_o, t_1); by definition of t_o, S must be non-null. If $\dot{y}_j(t) \geq 0$ over (t_o, t_1), then $y_j(t) \geq 0$ over the same interval by (18). From the definition of S,

$$\dot{y}_s(t) < 0 \text{ in } (t_o, t_1), \tag{20}$$

where the subscript S means the sub-vector composed of components of S. Let B be the principal minor of A with rows and columns in S, C the minor of A with rows in S and columns not in S.

From the hypothesis as to the differential equation satisfied by $y(t)$ and (20), we conclude that,

(21) $\dot{y}_S = By_S + Cy_{\bar{S}} + b_S$ for t in (t_o, t_1),

where $y_{\bar{S}}$ contains all components of y not in S. Since A is a Metzler matrix, C contains only non-negative components. By definition of S, $y_{\bar{S}} \geq 0$; hence $Cy_{\bar{S}} \geq 0$. By hypothesis, $b_S \geq 0$. Therefore, from (21),

(22) $By_S < 0$.

From Lemma 1, B is a stable Metzler matrix; by Lemma 2, then, $y_S \leq 0$ for t in (t_o, t_1). But this contradicts the definition of S. Hence, the supposition that the set of non-negative t-values for which $y_j(t) < 0$ for some j is non-null has led to a contradiction, and the theorem is proved.

4. The Adjustment Process with Steadily Increasing Demands

We shall now examine the process of adjustment described in sections 1 and 2 when the demand is shifting steadily upwards in time on some or all of the interrelated markets. For simplicity, we assume that the supply curve is not changing; however, the following analysis would remain valid if the supply were also shifting upwards in time but not more rapidly than the demand. We will also assume that, to begin with, supply and demand are equal.

We shall then continue to assume that (2) holds, but (5) becomes,

(23) $D_i = \sum_{j=1}^{n} b_{ij} p_j + d_i + e_i t,\ b_{11} > 0,\ b_{ij} \geq 0$ for $i \neq j$, $e_i \geq 0$.

The definition of the shortage X_i remains as before, and the adjustment of prices continues to be described by (7). Then the following discussion remains valid with slight modification. A term $e_i t$ is added on the right side of (8) and, equivalently, of (10). Let $\underline{e}$ be the vector whose components are e_i. Then, in vector notation, (13) remains valid, while (14) becomes,

(24) $x = Mp + n + et.$

M still satisfies (11). Further, we will assume that the system defined by (13) and (24) would be stable in the absence of trends, i.e., if $e = 0$, so that (16) remains valid. The characteristic roots of MK are the same as those of $K^{-1}(MK)K = KM$, so that the latter is also stable; it is clearly a Metzler matrix, since it is derived from M by multiplying each column by a positive constant.

(25) KM is a stable Metzler matrix.

Differentiate (24) with respect to time.

(26) $\dot{x} = M\dot{p} + e,$

where, it will be recalled, dots denote differentiation with respect to time. Substitute for $\dot{p}$ from (13) into (26);

(27) $\dot{x} = MK\,x + e$

From (25), we see that the solution $x(T)$ of the differential equation (27) converges to a limit, which must be such that $\dot{x} = 0$.

(28) $\lim_{t \to \infty} x(t) = -K^{-1}M^{-1}e.$

The assumption that supply and demand are equal at the beginning can be expressed by saying that,

(29) $x(0) = 0,$

where $t = 0$ is taken as the beginning of the process. Let $t = 0$ in (27); from (29) and (23),

(30) $\dot{x}(0) = e \geq 0.$

Differentiate (27) with respect to time.

(31) $d\dot{x}/dt = MK\dot{x}.$

We can apply the Theorem with $\underline{y}$ replaced by $\dot{x}$, $\underline{b}$ by 0, and A by MK; the hypotheses are satisfied, according to (25), (30), and (31), so that,

(32) $\dot{x}(t) \geq 0$ for $t \geq 0.$

Thus the shortage on each market increases from the initial value of 0 towards the asymptotic limit given by (28). In particular,

(33) $x(t) \geq 0$ for $t \geq 0.$

From (13) and (33),

(34) $\dot{p}(t) \geq 0$ for $t \geq 0,$

so that the price on each market is increasing over time. Differentiate (13) with respect to time.

(35) $d\dot{p}/dt = K\dot{x}.$

Substitute for $\dot{x}$ from (26).

(3') $\quad d\dot{p}/dt = KM\dot{p} + Ke.$

From (1') and (3'), it follows that $\dot{p}$ converges to a limit as t approaches infinity.

(37) $\quad \lim_{t\to\infty} \dot{p}(t) = -(KM)^{-1}Ke = -M^{-1}e.$

From (34) and (37), we conclude that prices rise on all markets and the increase approaches a constant rate which will usually be positive on all markets, even those which do not themselves have an upward shift in demand; the limiting rate of price increase depends only on supply and demand conditions and is independent of the speeds of adjustment.

At any time t, let p^* be the vector of prices which would clear the market, i.e., make the shortage zero. From (24),

(38) $\quad 0 = Mp^* + cn + et.$

Multiply through in (38) by K.

(39) $\quad 0 = KMp^* + Kn + Ket.$

Substitute from (24) into (13).

(40) $\quad \dot{p} = KMp + Kn + Ket.$

Finally, let q be the difference between the market-clearing price p^* and the actual price p, i.e., $p^* - p$. Subtract (40) from (39),

(41) $\quad -\dot{p} = KMq,$

or,

(42) $\quad q = -(KM)^{-1}\dot{p}.$

From (16) and Lemma 3, $-(KM)^{-1}$ is a non-negative matrix. Then from (34),

(43) $q(t) \geq 0$ for all $t \geq 0$,

while from (42) and (37),

(44) $\lim_{t \to \infty} q(t) = (KM)^{-1}M^{-1}e = M^{-1}K^{-1}M^{-1}e$.

Thus, the actual price is always below the price which would clear the market, the difference approaching a limit which is less the faster the speed of reaction on the different markets. It can also be shown that the difference between the actual and the market-clearing prices widens as time goes on.